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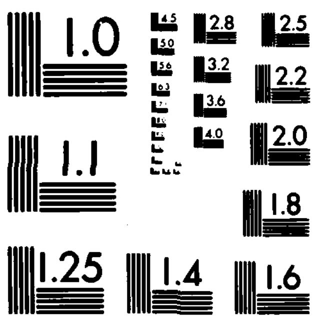

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MRC Technical Summary Report #2597

LINEAR COMPLEMENTARITY PROBLEMS SOLVABLE BY AN EFFICIENT POLYNOMIALLY BOUNDED PIVOTING ALGORITHM

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November 1983

(Received September 29, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

Applied to two important classes of linear complementarity problems defined by an $n \times n$ matrix, the parametric principal pivoting algorithm, using a suitably chosen (and easily computable) parametric vector, terminates with a desired solution after at most n pivot operations. Since each pivot can be performed using at most $\overset{\text{sq } n}{0(n^2)}$ arithmetic operations, the total computational complexity of the algorithm for solving these linear complementarity problems is no more than $\overset{\text{cu } m}{0(n^3)}$. In one of the two classes of problems being studied, the complexity is $\overset{\text{sq } n}{0(n^2)}$ because the matrix involved is 5-diagonal which allows each pivot to be performed in linear time. Some discussion in connection with Lemke's well-known almost complementary pivoting algorithm is also addressed.

AMS (MOS) Subject Classification: 90C20, 90C33

Key Words: Linear complementarity, parametric principal pivoting, efficient algorithm, polynomially bounded, concave regression, hidden Z, diagonally dominant

Work Unit Number 5 - Optimization and Large Scale Systems

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Many well-known pivoting methods for solving the linear complementarity problem have been shown to exhibit an exponential computational complexity. In this paper, we identify two classes of linear complementarity problems which can be solved by a numerically efficient as well as polynomially bounded pivoting algorithm. One of these classes of problems arises from the concave regression problem which has practical applications in statistics.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LINEAR COMPLEMENTARITY PROBLEMS SOLVABLE BY AN EFFICIENT POLYNOMIALLY BOUNDED PIVOTING ALGORITHM

Jong-Shi Pang*

1. INTRODUCTION.

It is known that the general linear complementarity problem defined by an arbitrary matrix is NP-complete [5]. Hence, it is unlikely that there will be a polynomially bounded algorithm for solving an arbitrary linear complementarity problem. With respect to several most notable pivoting methods, examples of problems have been constructed which show that these methods can require *an exponential number of pivots* [2, 12, 22]. These cited studies are all theoretical in nature and provide the worst-case analysis of the linear complementarity problem. From a practical point of view, it is more desirable to be able to identify classes of problems (with applications) which are solvable by an algorithm that is both numerically efficient as well as polynomially bounded. Except for the trivial ones (e.g. those defined by triangular matrices) the class of linear complementarity problems with a Z-matrix is perhaps the best known member belonging to such a category [4, 30]. (Another related class can be found in [25].) These latter complementarity problems have applications in the numerical solution of free boundary value problems, optimal stopping, isotonic regression and others.

In this paper, we identify two classes of linear complementarity problems and show that with a suitably chosen (and easily computable) parametric vector, the parametric principal pivoting algorithm [6, 27] will compute a solution after at most n pivot iterations where n is the order of the matrix defining the complementarity problems. One such class arises from the concave regression problem which has practical applications in statistical regression analysis. The latter problem was first formulated by Hildreth [13] for the estimation of marginal productivity curves and is concerned, in general, with finding a least-square estimate of a certain functional relationship between some dependent

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and independent variables which is known to be concave. Recent references on the concave regression problem include [11, 14, 31]. The resulting class of linear complementarity problems is defined by a 5-diagonal symmetric positive definite but non-Z matrix. The 5-diagonal structure greatly simplifies the practical implementation of the parametric principal pivoting algorithm which results in an efficient $0(n^2)$ method for solving the concave regression problem.

The primary reason why a great deal of emphasis is placed on the concave regression problem is because it is this practical problem which has challenged us to develop an algorithm that is both numerically efficient and polynomially bounded and has subsequently led us into this entire study. The desire to derive such a fast algorithm is in turn motivated by a close relative of the concave regression problem, namely, the isotonic regression problem. This latter problem has been well studied in statistics [1] and a linear-time algorithm has been developed [17, 31]. From a complementarity point of view, the isotonic regression problem can be formulated as a linear complementarity problem with a tridiagonal Stieltjes matrix and thus its solution by a linear-time algorithm is to be expected [4, 9, 10]. On the other hand, as we shall see, the matrix defining the linear complementarity problem arising from the concave regression problem is not Z.

The other class of linear complementarity problems identified in this paper consists of those defined by a matrix whose transpose is hidden Minkowski. A hidden Minkowski matrix is a P-matrix which is hidden Z. The class of hidden Z-matrices was introduced by Mangasarian [20] in connection with the study of solving linear complementarity problems as linear programs (see also [8]). The name "hidden Z" was coined because of the relation to a hidden Leontief matrix [23]. Specifically, a matrix M is hidden Z if there exist Z-matrices X and Y such that the two conditions below are satisfied

$$MX = Y \tag{1}$$

$$r^T X + s^T Y > 0 \quad \text{for some} \quad r,s \geq 0 \; . \tag{2}$$

Obviously, a Z-matrix is hidden Z. Many basic properties of a hidden Minkowski matrix have been obtained in [24]. In particular, it is known that an H-matrix with positive diagonals is hidden Minkowski. (A matrix M is H if its comparison matrix $\hat{M}$ defined by

$$\bar{M}_{ij} = \begin{cases} -|M_{ij}| & \text{if } i \neq j \\ |M_{ii}| & \text{if } i = j \end{cases}$$

is P.) Examples of H-matrices include the column (or row) strictly diagonally dominant matrices and of course the Minkowski matrices. Other examples of hidden Minkowski matrices can be found in [24]. Since the transpose of an H-matrix is obviously an H-matrix, it follows from our analysis that a linear complementarity problem defined by an H-matrix with positive diagonals can be solved by an effective $O(n^3)$ pivoting algorithm. In particular, so can a problem with a strictly diagonally dominant matrix having positive diagonal entries.

Although Mangasarian [20] has shown that a linear complementarity problem with a hidden Z-matrix can be solved by a linear program (using an easily computable objective function), the only polynomially bounded algorithm for solving a general linear program [16] is known to be numerically inefficient [3]. The parametric principal pivoting algorithm, on the other hand, has been shown to exhibit good numerical performance even on problems of fairly large size [28, 29].

The question of efficiently identifying whether an arbitrary matrix is hidden Z reamins unsolved. (Basically, the difficulty has to do with the nonlinearity of the second defining condition (2).) However, that of checking if a matrix is hidden Minkowski can be effectively answered by solving two linear programs [26]. The procedure described in the cited reference will find the two Z-matrices X and Y satisfying (1) and (2) if M is indeed hidden Minkowski. For certain subclasses (like the H-matrices with positive diagonals), the matrices X and Y can be obtained easily without solving any linear program (see [8, 20, 24] for more such subclasses). Related to this discussion is the open question of whether the transpose of a hidden Z-matrix is hidden Z.

The organization of the remainder of this paper is as follows. In the next section, we give a quick review of the parametric principal pivoting algorithm for solving a linear complementarity problem with an $n \times n$ P-matrix and state a general (sufficient) condition under which the algorithm will terminate with the desired solution after at most n pivots. We also discuss an efficient implementation of the algorithm when the condition is satisfied and establish its $O(n^3)$ computational complexity. Section 3 deals with the concave regression problem. We show how this practical problem can be formulated as a special linear complementarity problem and demonstrate that the condition guaranteeing the linear termination of the parametric principal pivoting algorithm is indeed satisfied by this special problem. In Section 4, we study a general linear complementarity problem defined by a P-matrix whose transpose is hidden Z and show that the same sufficient condition on the parametric principal pivoting algorithm is also satisfied by this class of problems. Finally, in the fifth and last section, we extend our discussion to Lemke's almost complementary pivoting algorithm [18, 19] and show that the same condition given in Section 1 is also sufficient for Lemke's method to terminate in at most $n + 1$ pivots with a desired solution when applied to a general linear complementarity problem with an $n \times n$ nondegenerate matrix.

2. THE PARAMETRIC PRINCIPAL PIVOTING ALGORITHM

We find it useful to quickly review the parametric principal pivoting algorithm in terms of its practical implementation [6, 27]. Given the linear complementarity problem (LCP)

$$y = q + Mx \geq 0, \quad x \geq 0, \quad y^T x = 0 \tag{3}$$

where the matrix M is P, we augment it by a parametric vector p and consider the parametric LCP

$$y = q + \theta p + Mx \geq 0, \quad x \geq 0, \quad y^T x = 0$$

where θ is a parameter to be driven to zero. The vector p is chosen positive. Assume that several (principal) pivots have been performed. Let $L(K)$ denote the currently basic (nonbasic) x-variables. (Initially, $L = \phi$.) With respect to these index sets, the canonical tableau may be written in the form

$$\begin{array}{r|c|c|cc|} & & \theta & y_L & x_K \\ \hline x_L = & \bar{q}_L & \bar{p}_L & M_{LL}^{-1} & -M_{LL}^{-1}M_{LK} \\ y_K = & \bar{q}_K & \bar{p}_K & M_{KL}M_{LL}^{-1} & (M/M_{LL}) \\ \hline \end{array}$$

where $(\bar{q}_L, \bar{p}_L)$ is the (unique) solution to the system of linear equations

$$M_{LL}(\bar{q}_L, \bar{p}_L) = -(q_L, p_L)$$

and

$$(\bar{q}_K, \bar{p}_K) = (q_K, p_K) + M_{KL}(\bar{q}_L, \bar{p}_L) ,$$

and where (M/M_{LL}) denotes the Schur complement of M_{LL} in M:

$$(M/M_{LL}) = M_{KK} - M_{KL}M_{LL}^{-1}M_{LK} .$$

(See [7] for various properties of the Schur complement.)

To determine the next pivot, the ratio test is performed:

$$\bar{\theta} = \max\{-\bar{q}_i/\bar{p}_i : \bar{p}_i > 0\} . \tag{4}$$

If $\bar{p} \leq 0$ or $\bar{\theta} \leq 0$, then the desired solution to the original LCP(3) is obtained as

$$x^* = (\bar{q}_L, 0) .$$

Otherwise, let k be a maximizing index in (4). Update the index set L (and its complement K) by the rule:

$$L_{new} = \begin{cases} L_{old} \setminus \{k\} & \text{if } k \in L_{old} \\ L_{old} \cup \{k\} & \text{if } k \notin L_{old} . \end{cases}$$

This completes one (pivot) iteration of the algorithm.

Now suppose that the vector p is chosen so that

$$M_{LL}^{-1} p_L \geqq 0 \quad \text{for all } L . \tag{5}$$

Then the maximizing index k can never occur in L_{old}. Thus the cardinality of the set L is increasing by one at each pivot. Consequently, unless the algorithm has already terminated (with a desired solution), it will continue until the complement K reaches empty, at which point, we have $\bar{p} = -M^{-1}p \leq 0$ and the algorithm terminates. We have therefore proved

Theorem 1. Let M be a P-matrix of order n. If there exists a vector $p > 0$ such that condition (5) holds, then the parametric principal pivoting algorithm, using p as the parametric vector, will compute a solution to the LCP(3) in at most n pivots.

Remarks. (i) No nondegeneracy assumption is needed in Theorem 1.

(ii) Underlying the proof of Theorem 1 is the key idea that once an x-variable becomes basic, it will *stay basic until termination.* Loosely stated, Theorem 1 asserts that all the basic x-variables can be identified in no more than n pivot steps. Of course, once those variables are determined, the desired solution to (3) is readily obtained.

Based on the idea pointed out in Remark (ii) above, it is possible to simplify the implementation of the parametric principal pivoting algorithm. Indeed, since a basic x-variable can not become nonbasic again, we need to keep track of the nonbasic components $(\bar{q}_K, \bar{p}_K)$ only and restrict the ratio comparisons (4) to such components. Moreover, exploiting the fact that the index set K decreases by one element k (the maximizing index) at each pivot, we may update these nonbasic components by the following recursive formula which does not require the knowledge of the basic components:

$$(\bar{q}_{K'}, \bar{p}_{K'})_{new} = (\bar{q}_{K'}, \bar{p}_{K'})_{old} + (M_{K'k} - M_{K'L}\bar{M}_{Lk})(\bar{q}_k, \bar{p}_k) \tag{6i}$$

where $K' = K\setminus\{k\}$ and $\bar{M}_{Lk}$ is the solution to the system of linear equations

$$M_{LL}\bar{M}_{Lk} = M_{Lk} \ . \tag{6ii}$$

(The formulas (6i) and (6ii) are easy to verify.)

Summarizing the discussion, we give below a step-by-step implementation of the parametric principal pivoting algorithm for solving the LCP(3), assuming that a vector p has been chosen as specified in Theorem 1.

Step 0. (Initialization) Set $L = \phi$ and $K = \{1,\ldots,n\}$. Set $\bar{p} = p$ and $\bar{q} = q$.

Step 1. (Termination Test) Determine the critical value

$$\bar{\theta} = \max\{-\bar{q}_i/\bar{p}_i : i \in K \text{ and } \bar{p}_i > 0\} \ . \tag{4)'$$

If $\bar{p}_K \leq 0$ or $\bar{\theta} \leq 0$, go to Step 3. Otherwise, let $k \in K$ be a maximizing index.

Step 2. (Update of the Nonbasic Components) If $K = \{k\}$, set $K = \phi$ and $L = \{1,\ldots,n\}$, go to Step 3. Otherwise, solve the system of linear equations (6ii) and compute (6i). Set $K = K'$ and $L = L \cup \{k\}$. Go to Step 1.

Step 3. (Output of Solution) Solve the system of linear equations for x_L^*:

$$M_{LL}x_L^* = -q_L \ .$$

The vector $x^* = (x_L^*,0)$ is the unique solution to the LCP (3). Stop

The success of the above algorithm clearly hinges on the ability to find the crucial vector p. Continuing to assume that it is available, we analyze the complexity of the algorithm. Step 1 requires $0(n)$ comparisons. By using an adaptive matrix factorization (such as LU) scheme, the system (6ii) can be solved in $0(n^2)$ time, so can the one in Step 3. The computations in (6i) can be achieved in the same amount of time. Since the algorithm will terminate after at most n passes through Steps 1 and 2 and since the storage requirement is obviously no more than $0(n^2)$, the overall complexity becomes $0(n^3)$. If the matrix M is banded with small width (such as the one to be analyzed in the next section), both Steps 2 and 3 can be carried out in linear time (even without any adaptive procedure). In this case, the complexity reduces to $0(n^2)$.

Remarks. (i) The above complexity analysis assumes that the systems of linear equations (6ii) are solved by a direct method (such as Gaussian elimination). For problems of very large size (say when n is a few thousands), it may be more advantageous to solve such systems by iterative methods (like SOR). In this case, the analysis would not be applicable because of the infinite nature of the iterative schemes.

(ii) The implementation given above ignores the basic components $(\bar{q}_L, \bar{p}_L)$ until termination actually occurs. If for some reason they are needed, they can be obtained recursively from the following formulas:

$$(\bar{q}_L, \bar{p}_L)_{new} = (\bar{q}_L, \bar{p}_L)_{old} - M_{Lk}(\bar{q}_k, \bar{p}_k)_{new}$$

$$(\bar{q}_k, \bar{p}_k)_{new} = -(\bar{q}_k, \bar{p}_k)_{old}/(M_{kk} - M_{kL}\bar{M}_{Lk}) .$$

3. THE CONCAVE REGRESSION PROBLEM.

In this section, we study the concave regression problem [31] and show that when it is formulated as the LCP(3), the vector p of all ones will satisfy the required properties in Theorem 1. Thus, the analysis of Section 2 applies and we obtain an effective $O(n^2)$ algorithm for solving the concave regression problem.

Given an integer $n \geq 1$, data points $\{a_i\}_{i=1}^{n+2}$, scalars $\{\alpha_i\}_{i=1}^{n+2}$ with $\alpha_i < \alpha_{i+1}$ $(i = 1,\dots,n+1)$ and weights $\{w_i\}_{i=1}^{n+2}$ with $w_i > 0$ $(i = 1,\dots,n+2)$, the concave regression problem is to find numbers $\{u_i\}_{i=1}^{n+2}$ to

$$\text{minimize} \sum_{i=1}^{n+2} w_i(u_i - a_i)^2 \tag{7}$$

subject to

$$\frac{u_{i+1} - u_i}{\alpha_{i+1} - \alpha_i} \leq \frac{u_i - u_{i-1}}{\alpha_i - \alpha_{i-1}} \quad (i = 2,\dots,n+1)\ .$$

The constraints above express the fact that the consecutive slopes of the line segments joining the points $\{(\alpha_i,u_i)\}_{i=1}^{n+2}$ (in the plane) are non-increasing, i.e. the piecewise-linear curve connecting those $n + 2$ points is concave.

To write (7) in matrix form, let W be an $m \times m$ diagonal matrix with weights $\{w_i\}_{i=1}^{m}$ as the diagonal entries $(m = n + 2)$. Define the positive scalars

$$\beta_i = 1/(\alpha_{i+1} - \alpha_i) \quad i = 1,\dots,n+1$$

and let

$$A = \begin{bmatrix} -\beta_1 & \beta_1 + \beta_2 & -\beta_2 & & & \\ & -\beta_2 & \beta_2 + \beta_3 & -\beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\beta_n & \beta_n + \beta_{n+1} & -\beta_{n+1} \end{bmatrix}$$

be the $n \times m$ matrix of the concavity constraints in (7). We may now restate problem (7) as to find a vector $u \in R^m$ to

$$\text{minimize} \quad \tfrac{1}{2} u^T W u - u^T W a \tag{7)'}$$

$$\text{subject to}$$

$$Au \geqq 0 .$$

The Karush-Kuhn-Tucker conditions of (7)' are

$$0 = -a + u - W^{-1}A^T x$$

$$y = Au \geqq 0, \quad x \geqq 0, \quad y^t x = 0 .$$

Eliminating the vector u, we obtain the LCP

$$y = Aa + AW^{-1}A^T x \geqq 0, \quad x \geqq 0, \quad y^T x = 0 . \tag{8}$$

Obviously, if x^* solves the above LCP, then the vector

$$u^* = a + W^{-1}A^T x^*$$

solves the concave regression problem (7).

It is easy to see that the matrix A has full row rank. Thus the matrix

$$M = AW^{-1}A^T \tag{8)'}$$

defining the LCP(8) is symmetric positive definite. Moreover, M is a 5-diagonal matrix (i.e. a band matrix of width 5) and its entries have the sign pattern

$$\begin{bmatrix} + & - & + & & & & & \\ - & + & - & + & & & & \\ + & - & + & - & + & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & + & - & + & - & + \\ & & & & + & - & + & - \\ & & & & & + & - & + \end{bmatrix} .$$

Thus M is not Z. (We do not know if M is hidden Z or not.)

<u>Theorem 2</u>. Let M be given by (8)'. Then condition (5) holds if p is chosen as the vector of all ones. Consequently, the conclusion of Theorem 1 applies to the LCP(8).

The proof of Theorem 2 is by induction on the cardinality of the index set L. For this purpose, we derive some properties of the matrix M defined above.

Since M is completely determined by the scalars $\{\beta_i, w_i\}$, we shall say that M is of type $(\{\beta_i\}_{i=1}^{n+1}, \{w_i\}_{i=1}^{n+2})$. We say that a matrix M' is of the same type as M if M' is defined by some positive scalars $\{\beta_i'\}_{i=1}^{n'+1}$ and $\{w_i'\}_{i=1}^{n'+2}$ in the same way as M is by $\{\beta_i\}_{i=1}^{n+1}$ and $\{w_i\}_{i=1}^{n+2}$. Obviously, if M_{LL} is a principal submatrix of M consisting of consecutive rows and columns, then M_{LL} if of the same type as M. Indeed, if

$$L = \{j, j+1, \ldots, j+k\} ,$$

then M_{LL} is of type $(\{\beta_i\}_{i=j}^{j+k+1}, \{w_i\}_{i=j}^{j+k+2})$.

By the k-th forward (backward) leading principal submatrix of M, we mean the submatrix consisting of the first (last) k rows and k columns. The following few lemmas establish certain invariance properties of the Schur complements of leading principal submatrices in M. They are all stated in terms of forward leading principal submatrices. Similar results hold for backward leading principal submatrices. The first lemma says that the Schur complement of M_{11} in M is of the same type as M. Moreover, the scalars defining this Schur complement are not much different from those defining M and can be obtained easily from some simple expressions.

<u>Lemma 1</u>. The Schur complement (M/M_{11}) is of type

$$(\{\tilde{\beta}_2, \{\beta_i\}_{i=3}^{n+1}\}, \{\tilde{w}_2, \tilde{w}_3, \{w_i\}_{i=4}^{n+2}\})$$

where

$$\tilde{\beta}_2 = \beta_1\beta_2(w_1^{-1}\beta_1 + w_2^{-1}(\beta_1 + \beta_2))/(w_1^{-1}\beta_1^2 + w_2^{-1}(\beta_1 + \beta_2)^2)$$

$$\tilde{w}_2 = M_{11}/(w_2^{-1}(w_1^{-1}\beta_1^2 + w_3^{-1}\tilde{\beta}_2(\beta_1 + \beta_2)))$$

$$\tilde{w}_3 = M_{11}/(w_3^{-1}(w_1^{-1}\beta_1^2 + w_2^{-1}(\beta_1 + \beta_2)^2)) .$$

<u>Proof</u>. This follows from a straightforward computation of (M/M_{11}).

Generalizing Lemma 1, we have

<u>Lemma 2</u>. Let N_k denote the k-th forward leading principal submatrix of M. Then the Schur complment (M/N_k) is of type

$$(\{\tilde{\beta}_{k+1}, \{\beta_i\}_{i=k+2}^{n+1}\}, \{\tilde{w}_{k+1}, \tilde{w}_{k+2}, \{w_i\}_{i=k+3}^{n+2}\})$$

for some suitable scalars $\tilde{\beta}_{k+1}$, $\tilde{w}_{k+1}$ and $\tilde{w}_{k+2}$.

<u>Proof</u>. This can be proved by induction, utilizing Lemma 1 and the fact that

$$(M/N_k) = (\tilde{M}/\tilde{N}_{k-1})$$

where $\tilde{M} = (M/M_{11})$ and $\tilde{N}_{k-1}$ denotes the (k-1)-st leading principal submatrix of $\tilde{M}$.

Lemma 3. Let M_{LL} be a principal submatrix of M. Let N_k be the k-th forward leading principal submatrix of M_{LL}. Then the Schur complement (M_{LL}/N_k) is a principal submatrix of some matrix M' of the same type as M.

Proof. By means of an inductive argument, like the one used in Lemma 2, it suffices to prove for $k = 1$. If M_{LL} consists of consecutive rows and columns from M, then M_{LL} is itself a matrix of the same type as M and thus the conclusion follows from Lemma 1. On the other hand, if some rows (and columns) in M_{LL} are not consecutive, we may fill in those missing rows and columns to get a larger principal submatrix $M_{L'L'}$ with $L \subsetneq L'$. Then $(M_{LL}/(M_{LL})_{11})$ is a principal submatrix of $(M_{L'L'}/(M_{LL})_{11})$ which by Lemma 1 again, is a matrix of the same type as M.

The following lemma is easy to see. It allows us to perform some scaling operation in the main inductive proof of Theorem 2.

Lemma 4. Let N be obtained from the matrix M by deleting its second row and second column and then scaling the first row and the first column by a positive scalar δ (the (1,1)-entry is thus scaled by δ^2). Then N is equal to the principal submatrix obtained from the matrix M' by deleting its second row and second column where M' is of type

$$(\{\delta\beta_1, \delta\beta_2, \{\beta_i\}_{i=3}^{n+1}\}, \{w_i\}_{i=1}^{n+2}) .$$

Remark. We should point out that if we perform the same scaling operation to the matrix M itself, the resulting matrix will not be of the same type as M.

In [31], it was proved that if the set of data points $\{a_i\}_{i=1}^{n+2}$ is convex, then the least-square concave fit must be a straight line. This interesting geometric fact has an important algebraic consequence, namely, that the inverse of the matrix M is nonnegative. To see this, we translate the geometric statement in the context of the LCP(8), obtaining the implication:

$$Aa \leq 0 \implies y^* = Au^* = 0$$

which is equivalent to

$$Aa \leq 0 \implies x^* = -(AW^{-1}A^T)^{-1}Aa \geq 0 .$$

Since the matrix A has full row rank, the above implication is in turn equivalent to

$$y \geq 0 \implies (AW^{-1}A^T)^{-1}y \geq 0 .$$

thus $M^{-1} \geq 0$.

As a matter of fact, a stronger version of the above conclusion holds.

Lemma 5. Let $\{\beta_i\}_{i=1}^{n+1}$ and $\{w_i\}_{i=1}^{n+2}$ be positive scalars. Then a matrix M of type $(\{\beta_i\}_{i=1}^{n+1}, \{w_i\}_{i=1}^{n+2})$ has a positive inverse.

Notice that Lemma 5 does not follow from the preceding argument. In what follows, we give a proof based completely on matrix manipulations. By letting $\tilde{W}$ be the $n \times n$ diagonal matrix of the scalars $\{w_i\}_{i=2}^{n+1}$ and

$$B = \begin{bmatrix} \beta_2 + \beta_1 & -\beta_2 & & & \\ -\beta_2 & \beta_3 + \beta_2 & -\beta_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -\beta_{n-1} & \beta_n + \beta_{n-1} & -\beta_n \\ & & & -\beta_n & \beta_n + \beta_{n+1} \end{bmatrix}$$

be obtained from A by deleting its first and last columns, it is easy to see that

$$M = B\tilde{W}^{-1}B^T + w_1^{-1}\beta_1^2 e_1 e_1^T + w_{n+2}^{-1}\beta_{n+1}^2 e_n e_n^T \tag{9}$$

where e_1 and e_n are the first and last unit vectors in R^n. The above formula (9) identifies the matrix M as a simple rank-two update of $B\tilde{W}^{-1}B^T$. Notice that B is an irreducibly diagonally dominant Stieltjes matrix. Thus B has a nonnegative inverse, and so does $B\tilde{W}^{-1}B^T$. The next result gives an explicit inverse for B which is needed to prove Lemma 5.

Lemma 6. Let $m = n + 2$ and $\beta_i = 1/(\alpha_{i+1} - \alpha_i)$, $i = 1,\ldots,n + 1$. Then

$$B^{-1} = \frac{1}{\alpha_m - \alpha_1} \begin{bmatrix} (\alpha_2 - \alpha_1)(\alpha_m - \alpha_2) & (\alpha_2 - \alpha_1)(\alpha_m - \alpha_3) & \cdots & (\alpha_2 - \alpha_1)(\alpha_m - \alpha_{m-1}) \\ (\alpha_2 - \alpha_1)(\alpha_m - a_3) & (\alpha_3 - \alpha_1)(\alpha_m - \alpha_3) & \cdots & (\alpha_3 - \alpha_1)(\alpha_m - \alpha_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_2 - \alpha_1)(\alpha_m - \alpha_{m-1}) & (\alpha_3 - \alpha_1)(\alpha_m - \alpha_{m-1}) & \cdots & (\alpha_{m-1} - \alpha_1)(\alpha_m - \alpha_{m-1}) \end{bmatrix}$$

Proof. This can be verified by directly multiplying out BB^{-1}.

Remark. Lemma 6 is believed to have its own interest. For example, it provides an explicit inverse for the matrix

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

which occurs very often in the discretization of elliptic partial differential equations.

In order to establish Lemma 5, we first show that a certain entry of M^{-1} is positive.

Lemma 7. $(M^{-1})_{1,n} > 0$.

Proof. Applying the Sherman-Morrison-Woodbury formula [15, p. 124] to (9), we obtain

$$M^{-1} = (B\tilde{W}^{-1}B^T)^{-1} - (w_1^{-1/2}\beta_1\bar{e}_1^T, w_m^{-1/2}\beta_{n+1}\bar{e}_n^T)$$

$$\left[I_{2\times 2} + \begin{pmatrix} w_1^{-1/2}\beta_1 e_1^T \\ w_m^{-1/2}\beta_{n+1}e_n^T \end{pmatrix} (B\tilde{W}^{-1}B^T)^{-1}(w_1^{-1/2}\beta_1 e_1, w_m^{-1/2}\beta_{n+1}e_n)\right] \begin{pmatrix} w_1^{-1/2}\beta_1\bar{e}_1 \\ w_m^{-1/2}\beta_{n+1}\bar{e}_n \end{pmatrix}$$

where

$$(\bar{e}_1, \bar{e}_n) = (B\tilde{W}^{-1}B^T)^{-1}(e_1, e_n) .$$

Define three scalars

$$\gamma = \sum_{k=2}^{m-1} w_k(\alpha_k - \alpha_1)(\alpha_m - \alpha_k), \ \delta = \sum_{k=2}^{m-1} w_k(\alpha_m - \alpha_k)^2 \ \text{ and } \ \sigma = \sum_{k=2}^{m-1} w_k(\alpha_k - \alpha_1)^2 .$$

Then, by Lemma 6 and an easy manipulation, we obtain

$$(M^{-1})_{1,n} = e_1^T M^{-1} e_n = \frac{(\alpha_2 - \alpha_1)(\alpha_m - \alpha_{m-1})}{(\alpha_m - \alpha_1)^2}\left(\gamma - (\delta,\gamma)G^{-1}\binom{\gamma}{\sigma}\right)$$

where G is the 2×2 positive definite matrix

$$G = \begin{bmatrix} \delta + w_1(\alpha_m - \alpha_1)^2 & \gamma \\ \gamma & \sigma + w_m(\alpha_m - \alpha_1)^2 \end{bmatrix} .$$

By the Schur determinantal formula (see [7]), we deduce

$$(M^{-1})_{1,n} = \frac{(\alpha_2 - \alpha_1)(\alpha_m - \alpha_{m-1})}{(\alpha_m - \alpha_1)^2} \frac{\det R}{\det G}$$

where R is the 3×3 matrix

$$R = \begin{bmatrix} \gamma & \delta & \gamma \\ \gamma & & \\ \sigma & G & \end{bmatrix} .$$

Obviously,

$$\det R = w_1 w_m (\alpha_m - \alpha_1)^4 \gamma > 0 .$$

Thus, $(M^{-1})_{1,n} > 0$ as desired.

<u>Proof of Lemma 5</u>. We use induction on n, which is the order of the matrix M. It is obvious that the assertion is true for $n = 1$. Suppose that it is true for $k < n$. Let M be a matrix of type $(\{\beta_i\}_{i=1}^{n+1}, \{w_i\}_{i=1}^{n+2})$. We may write

$$M = \begin{pmatrix} M_{11} & a^T \\ a & N \end{pmatrix}$$

where N is the $(n-1)$-st backward leading principal submatrix of M. According to Lemma 1, the Schur complement (M/M_{11}) is of the same type as M. Thus, the induction hypothesis implies that $\bar{N} = (M/M_{11})^{-1}$ is positive. We have (see [7] e.g.)

$$M^{-1} = \frac{1}{\delta} \begin{pmatrix} 1 & -a^T \bar{N}^{-1} \\ -\bar{N}^{-1} a & \delta \bar{N} \end{pmatrix}$$

where $\delta = (M/N)$. Since M is positive definite, $\delta > 0$. Consequently, with the possible exception of the second to last entries in the first row and first column, M^{-1} is positive. Similarly, applying the same argument to the partitioning

$$M = \begin{pmatrix} N' & b \\ b^T & M_{nn} \end{pmatrix}$$

where N' is the $(n-1)$-st forward leading principal submatrix of M, we can deduce that with the possible exception of $(M^{-1})_{1,n}$ and $(M^{-1})_{n,1}$, (these latter two entries are equal because M is symmetric) M^{-1} is positive. By Lemma 7, it thus follows that M^{-1} has all entries positive. This completes the proof.

We are now ready to prove the main Theorem 2.

<u>Proof of Theorem 2</u>. We use induction on the cardinality of L. The assertion is easily seen to hold for L consisting of no more than 2 indices. Suppose that the conclusion is true for all L consisting of no more than $k-1$ $(k \geq 3)$ indices. Let L be an arbitrary subset of $\{1,\ldots,n\}$ with k elements. We may partition the matrix M_{LL} in the following way:

$$M_{LL} = \begin{bmatrix} N_1 & Q_1^T & & & \\ Q_1 & N_2 & Q_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix}$$

where each N_i consists of consecutive rows and same columns of M and each Q_i is either identically zero or has all entries equal to zero except for the single one in the upper right corner which is positive. (In other words, we partition the set L into disjoint groups $\{L_i\}_{i=1}^{\ell}$ where indices in each L_i are consecutive.) If M_{LL} consists of just one single block N_1 (i.e. if the indices in L are all consecutive), then $M_{LL}^{-1}p_L > 0$ because M_{LL} is of the same type as M and Lemma 5 applies. So suppose that M_{LL} consists of more then one such block N_i.

If at least one Q_i is identically zero (i.e. if the last index in at least one group L_i differs by three or more from the first index in the immediately following group L_{i+1}), then the system of linear equations

$$M_{LL}\tilde{p}_L = p_L \tag{10}$$

decomposes into two smaller subsystems; therefore the positivity of $\tilde{p}_L$ follows from our induction hypothesis. So suppose that all Q_i have exactly one positive entry in the upper right corner (i.e. suppose that the last index in each group L_i is exactly two less than the first index in the immediately following group L_{i+1}).

Assume that the first leading block N_1 is or order $k' > 1$. We may write

$$N_1 = \begin{pmatrix} \bar{N}_1 & a \\ a^T & \delta \end{pmatrix}$$

where $\bar{N}_1$ is the $(k' - 1)$-st forward leading principal submatrix of N_1. Write

$$\tilde{p}_L = \begin{bmatrix} \tilde{p}_1 \\ \vdots \\ \tilde{p}_\ell \end{bmatrix}, \quad p_L = \begin{bmatrix} p_1 \\ \vdots \\ p_\ell \end{bmatrix}, \quad \tilde{p}_1 = \begin{pmatrix} \bar{p}_1 \\ r \end{pmatrix} \text{ and } p_1 = \begin{pmatrix} \hat{p}_1 \\ 1 \end{pmatrix}$$

according to the partitioning of M_{LL} and N_1. Pivoting on $\bar{N}_1$ in the system (10) yields the reduced system

$$\begin{bmatrix} \bar{\delta} & \bar{Q}_1^T & & & \\ \bar{Q}_1 & N_2 & Q_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix} \begin{bmatrix} r \\ \tilde{p}_2 \\ \vdots \\ \tilde{p}_{\ell-1} \\ \tilde{p}_\ell \end{bmatrix} = \begin{bmatrix} s \\ p_2 \\ \vdots \\ p_{\ell-1} \\ p_\ell \end{bmatrix} \qquad (11)$$

where $\bar{\delta} = (N_1/\bar{N}_1)$ and $\bar{Q}_1$ is the last column of Q_1. The scalar $s = 1 - a^T \bar{N}_1^{-1} \hat{p}_1$ is positive by induction hypothesis. Scaling the first row and first column in the system (11) by $1/s$, we obtain

$$\begin{bmatrix} \frac{1}{s^2} \bar{\delta} & \frac{1}{s} \bar{Q}_1^T & & & \\ \frac{1}{s} \bar{Q}_1 & N_2 & Q_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix} \begin{bmatrix} sr \\ \tilde{p}_2 \\ \vdots \\ \tilde{p}_{\ell-1} \\ \tilde{p}_\ell \end{bmatrix} = \begin{bmatrix} 1 \\ p_2 \\ \vdots \\ p_{\ell-1} \\ p_\ell \end{bmatrix} \qquad (12)$$

By Lemma 4, the matrix defining the system (12) is a principal submatrix of a matrix of the same type as M. Thus induction hypothesis implies that $r > 0$ and

$\tilde{p}_i > 0$ $(i = 2,\ldots,\ell)$. To show $\bar{p}_1 > 0$, we perform a backward principal pivot on the second to last blocks in the system (10) obtaining

$$\begin{pmatrix} \bar{N}_1 & a \\ a^T & \tilde{\delta} \end{pmatrix}\begin{pmatrix} \bar{p}_1 \\ r \end{pmatrix} = \begin{pmatrix} \hat{p}_1 \\ \tilde{s} \end{pmatrix} \tag{13}$$

where $\tilde{\delta}$ is the Schur complement of

$$\begin{bmatrix} N_2 & Q_2^T & & & \\ Q_2 & N_3 & Q_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix}$$

in the matrix

$$\begin{bmatrix} \delta & Q_1^T & & & \\ Q_1 & N_2 & Q_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix}$$

and the scalar $\tilde{s}$ is positive by induction hypothesis. By Lemma 3, it can be seen that the matrix defining the system (13) is of the same type as M and thus has a positive inverse (Lemma 5). Therefore $\bar{p}_1 > 0$.

Now suppose that the block N_1 consists of just one single entry. If the second block N_2 also consists of one single entry, then pivoting on N_1 in the system (10) gives the reduced system

$$\begin{bmatrix} \bar{N}_2 & Q_2^T & & & \\ Q_2 & N_3 & Q_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix} \begin{bmatrix} \tilde{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{p}_{\ell-1} \\ \tilde{p}_\ell \end{bmatrix} = \begin{bmatrix} \hat{p}_2 \\ p_3 \\ \cdot \\ \cdot \\ p_{\ell-1} \\ p_\ell \end{bmatrix}$$

where $\hat{p}_2 > 0$. Scaling the first row and first column of the system by $1/\hat{p}_2$, we obtain

$$\begin{bmatrix} \frac{1}{\hat{p}_2}\bar{N}_2 & \frac{1}{\hat{p}_2}Q_2^T & & & \\ \frac{1}{\hat{p}_2}Q_2 & N_3 & Q_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & Q_{\ell-2} & N_{\ell-1} & Q_{\ell-1}^T \\ & & & Q_{\ell-1} & N_\ell \end{bmatrix} \begin{bmatrix} \tilde{p}_2 \\ \tilde{p}_3 \\ \vdots \\ \tilde{p}_{\ell-1} \\ \tilde{p}_\ell \end{bmatrix} = \begin{bmatrix} 1 \\ p_3 \\ \vdots \\ p_{\ell-1} \\ p_\ell \end{bmatrix} \tag{14}$$

By Lemma 4 again, the matrix defining the above system is a principal submatrix of some matrix of the same type as M. Thus, the inductive hypothesis implies $\tilde{p}_i > 0$ $(i = 2,\ldots,\ell)$. To show $\tilde{p}_1 > 0$, perform a principal pivot on the third to last block $N_i (i = 3,\ldots,\ell)$ in the system (10) (since both N_1 and N_2 are single-entry blocks and L has $k \geq 3$ entires, there must be at least one block N_i $(i \geq 3)$), obtaining a 2×2 system from which the positivity of $\tilde{p}_1$ follows easily.

Finally, suppose that the second block N_2 is of order larger than 1. Then writing

$$N_2 = \begin{pmatrix} \bar{N}_2 & b \\ b^T & \delta' \end{pmatrix}$$

and applying an argument similar to the one used when N_1 is of order larger than 1, we can easily deduce that $\tilde{p}_L > 0$ as desired.

Summarizing, we always have $\tilde{p}_L > 0$ for any index set L. This completes the proof.

Remark. The reason why the scaling in the systems (12) and (14) is needed is because the induction hypothesis is applicable only when the right-hand vector in (12) and (14) consists of all ones.

4. THE CASE OF A HIDDEN MINKOWSKI MATRIX.

Returning to the general LCP(3), we prove

Theorem 3. Suppose that M^T is hidden Minkowski. Let X and Y be two Z-matrices satisfying

$$M^T X = Y \tag{15i}$$

$$r^T X + s^T Y > 0 \quad \text{for some vectors } r,s \geq 0 \ . \tag{15ii}$$

Then condition (5) holds for any vector $p > 0$ satisfying $X^T p > 0$. Consequently, the conclusion of Theorem 1 applies to the LCP(3).

Before proving Theorem 3, we compare it with the results obtained in [8,20] concerning the solvability of an LCP with a hidden Z-matrix as a linear program. Let M,X,Y and p be as given in Theorem 3. Then the (unique) solution to the LCP

$$y = q + M^T x \geq 0, \quad x \geq 0 \quad \text{and} \quad y^T x = 0 \tag{16}$$

can be obtained by solving the LP

$$\text{minimize} \quad p^T x$$

$$\text{subject to} \quad q + M^T x \geq 0 \quad \text{and} \quad x \geq 0 \ .$$

Theorem 3 implies that the same vector p can be used to start the parametric principal pivoting algorithm for solving the LCP

$$y = q + Mx \geq 0, \quad x \geq 0 \quad \text{and} \quad y^T x = 0 \tag{3}$$

and the algorithm will terminate after at most n pivots. Notice that (3) is defined by the matrix M whereas (16) is by its transpose M^T.

An example of a vector $p > 0$ satisfying $X^T p > 0$ is given by

$$p = r + Ms \tag{17}$$

where r and s satisfy (15ii), see [8]. More generally, the vector p computed by solving the system of linear equations

$$X^T p = e \tag{17'}$$

for any positive right-hand vector e will have the same required properties. Indeed, the fact that M^T is P implies that X^T is itself Minkowski (see Lemma 8 below). Hence the solution to (17)' is $p = X^{-T} e > 0$. In any case, p can be obtained in at most $0(n^3)$

time. Consequently, by the analysis of Section 2, the parametric principal pivoting algorithm will solve the LCP(3) in $O(n^3)$ time.

To prove Theorem 3, we quote the lemma below which summarizes two useful properties of a hidden Minkowski matrix. A proof of the lemma can be found in [24].

Lemma 8. Let M, X and Y be as given in Theorem 3. Then for any index set L,

(i) $M_{LL}^T(X/X_{KK}) = (W/X_{KK})$

where K is the complement of L and

$$W = \begin{pmatrix} Y_{LL} & Y_{LK} \\ X_{KL} & X_{KK} \end{pmatrix} .$$

(ii) The matrix W is Minkowski. (In particular, so are X and Y.)

Proof of Theorem 3. We first remark that if $X^Tp > 0$, then for any complementary index sets L and K,

$$(X^T/(X^T)_{KK})p_L > 0 .$$

From Lemma 8(i), we obtain

$$(M^T)_{LL} = (W/X_{KK})(X/X_{KK})^{-1}$$

which implies

$$(M_{LL})^{-1} = (W/X_{KK})^{-T}(X/X_{KK})^T$$

Thus, it follows that

$$(M_{LL})^{-1}p_L = (W/X_{KK})^{-T}(X/X_{KK})^Tp_L \geq 0$$

because (W/X_{KK}) is Minkowski (implied by Lemma 8(ii)) and thus has a nonnegative inverse, and $(X/X_{KK})^T = (X^T/(X^T)_{KK})$. This completes the proof.

The preceeding discussion shows that in general the parametric vector p can be calculated from either (17) or (17)'. In the Corollary below, we identify two classes of matrices M for which the vector p is available trivially.

Corollary 1. (i) If $M = Y^T + ab^T$ where Y is a Minkowski matrix and a and b are positive vectors, then $p = a$ satisfies condition (5).

(ii) If M has positive diagonal entries and is strictly row diagonally dominant, i.e.,

$$M_{ii} > \sum_{j \neq i} |M_{ij}| \quad \text{all } i \; ,$$

then the vector $p = (p_i)$ defined by

$$p_i = M_{ii} + \sum_{j:M_{ij}<0} M_{ij} \quad \text{all } i$$

satisfies condition (5).

Proof. It is known that a matrix M satisfying either (i) or (ii) is hidden Minkowski (see [8] e.g.). It is also obvious that the vector p in either case is positive. Thus it remains to verify that $X^T p > 0$ where X is as given in Theorem 3. In case (i), we have

$$M^T = Y(I + Y^{-1} ba^T) \; .$$

Thus

$$X^T = (I + Y^{-1} ba^T)^{-T} = I - \frac{ab^T Y^{-T}}{1 + b^T Y^{-T} a} \; .$$

Hence, it follows that

$$X^T a = a/(1 + b^T Y^{-T} a) > 0 \; .$$

In case (ii), we may write (see [8])

$$M = 2A - B$$

where $B = \bar{M}$ is the comparison matrix of M and $A = (M + \bar{M})/2$. It is easy to show that the required X^T is given by (see [8])

$$X^T = BA^{-1} \; .$$

With the vector p defined as specified, we easily deduce that $A^{-1}p$ is the vector of all ones. Thus $X^T p > 0$ because M is strictly row diagonally dominant.

Remark. The matrix M arising from the concave regression problem (cf. (8)') provides another example for which the parametric vector p is available trivially.

If M is an H-matrix with positive diagonals and if d is a positive vector such that $\bar{M}d > 0$, then the same proof of case (ii) in the above Corollary shows that the vector $p = (M + \bar{M})d/2$ satisfies condition (5).

5. ON LEMKE'S METHOD.

It is rather evident that the parametric principal pivoting algorithm presented in Section 2 is intimately related to Lemke's well-known almost complementary pivoting algorithm [18, 19]. Indeed, in [21], McCammon pointed out that Lemke's method can be implemented as a parametric pivot scheme in which the artificial variable is treated as a parameter (see also [19]). McCammon also discussed the relationship between this parametric version of Lemke's method and principal pivoting. As a result of such a connection, it is natural to ask whether the existence of a positive vector p satisfying condition (5) will imply a linear termination for Lemke's method applied to a LCP with a matrix which is not necessarily P. The result below gives an affirmative answer to this question. Recall that a matrix M is nondegenerate if all its principal submatrices are nonsingular.

<u>Theorem 4</u>. Let M be an $n \times n$ nondegenerate matrix. Suppose that there is a positive vector p satisfying condition (5). Then Lemke's almost complementary pivoting algorithm using p as the artificial vector, will terminate with a desired solution to the LCP(3) after at most $n + 1$ pivots.

<u>Proof</u>. Consider a current iteration (after the first) of Lemke's method. We have available an index set L corresponding to the basic x-variables and an index $t \notin L$ such that $\{y_t, x_t\}$ is the nonbasic pair. The canonical tableau may be written as (only the key entries are displayed)

$$\begin{array}{c|c|c|c|c|} & y_L & y_t & x_t & x_J \\ \hline \begin{matrix} x_L \\ \theta \end{matrix} = & & & -\begin{pmatrix} M_{LL} & p_L \\ M_{tL} & p_t \end{pmatrix}^{-1} \begin{pmatrix} M_{Lt} \\ M_{tt} \end{pmatrix} & \\ y_J & & & & \\ \hline \end{array} \tag{17}$$

where J is the complement of $\{t\} \cup L$.

Assume that y_t has just become nonbasic so that x_t is the incoming variable. We claim that condition (5) implies that the next pivot will not occur in an x_L-row. Indeed, after an easy computation, it is easy to verify that the (x_L, x_t)-entry in the above tableau is given more explicitly by

$$\left(\frac{M_{tt} - M_{tL}M_{LL}^{-1}M_{Lt}}{p_t - M_{tL}M_{LL}^{-1}p_L}\right) M_{LL}^{-1}p_L - M_{LL}^{-1}M_{Lt} . \tag{18}$$

(The fact that M is nondegenerate (implying M_{LL}^{-1} exists) and the nonsingularity of the basis matrix

$$B = \begin{pmatrix} M_{LL} & p_L \\ M_{tL} & p_t \end{pmatrix}$$

guarantee the nonvanishing of the denominator. The nonsingularity of the basis B is in turn guaranteed by the pivot operations.) By (5), the vector

$$\begin{pmatrix} \bar{p}_L \\ \bar{p}_t \end{pmatrix} = \begin{pmatrix} M_{LL} & M_{Lt} \\ M_{tL} & M_{tt} \end{pmatrix}^{-1} \begin{pmatrix} p_L \\ p_t \end{pmatrix} \geq 0 .$$

It is easy to show that

$$\bar{p}_L = M_{LL}^{-1}p_L - \left(\frac{p_t - M_{tL}M_{LL}^{-1}p_L}{M_{tt} - M_{tL}M_{LL}^{-1}M_{Lt}}\right) M_{LL}^{-1}M_{Lt}$$

$$\bar{p}_t = (p_t - M_{tL}M_{LL}^{-1}p_L)/(M_{tt} - M_{tL}M_{LL}^{-1}M_{Lt}) .$$

(Again, the nondegeneracy of M implies that the denominator in the above two expressions is nonzero.) Thus, it follows from (18) that the (x_L,x_t)-entry in tableau (17) is given by the vector $\bar{p}_L/\bar{p}_t$ which is nonnegative. Consequently, the increase of x_t will not decrease the values of the already-basic x_L-variables. Therefore, the next pivot will occur either in the θ-row (in which case a solution to the LCP(3) is obtained) or in a y_J-row (in which case the index set L increases by one element and the argument just given repeats itself).

Next, we show that the (θ,x_t)-entry in tableau (17) is negative. Indeed, this entry is $-1/\bar{p}_t$. Thus it is negative. Consequently, the increase of x_t is bounded above by θ. Hence, termination on a secondary ray can not occur.

Summarizing, we have proven that once an x-variable has become basic, it will stay basic until termination occurs. Since there are only n x-variables, the algorithm terminates after at most $n + 1$ pivots. Since termination on a ray is ruled out, a desired solution to the LCP(3) will be obtained. This establishes the theorem.

Remark. i) As in Theorem 1, no nondegeneracy assumption is needed to prove Theorem 4.

ii) The matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

satisfies the assumptions of Theorem 4 (with $p = (1,1)^T$). But it is not P.

Theorem 4 may be considered an extension of Theorem 1 to the case of a non-P matrix. Proofs to both theorems are based on the same key idea that an x-variable once becomes basic, stays basic. At present, it is not clear to us what class of matrices M (besides those considered in the last two sections) will produce an easily calculable vector p satisfying the required properties. We leave this as an open question for further investigation.

It is interesting to contrast Theorem 4 with the worst-case studies of [12, 22]. In these earlier studies, examples have been presented which demonstrate that Lemke's method can require an exponential number of pivots. On the other hand, Theorem 4 guarantees that for certain class of problems, the polynomial-time complexity of Lemke's method can be established. Although a complete knowledge of such problems is not yet available, the examples in Sections 3 and 4, together with those that are previously known should have amply demonstrated that the class is non-void and indeed contains some very interesting applications of the LCP.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2597	2. GOVT ACCESSION NO. AD-A136405	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) LINEAR COMPLEMENTARITY PROBLEMS SOLVABLE BY AN EFFICIENT POLYNOMIALLY BOUNDED PIVOTING ALGORITHM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jong-Shi Pang		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Optimization and Large Scale Systems
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE November 1983
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Linear complementarity, parametric principal pivoting, efficient algorithm, polynomially bounded, concave regression, hidden Z, diagonally dominant

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Applied to two important classes of linear complementarity problems defined by an $n \times n$ matrix, the parametric principal pivoting algorithm, using a suitably chosen (and easily computable) parametric vector, terminates with a desired solution after at most n pivot operations. Since each pivot can be performed using at most $O(n^2)$ arithmetic operations, the total

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ABSTRACT (cont.)

computational complexity of the algorithm for solving these linear complementarity problems is no more than $O(n^3)$. In one of the two classes of problems being studied, the complexity is $O(n^2)$ because the matrix involved is 5-diagonal which allows each pivot to be performed in linear time. Some discussion in connection with Lemke's well-known almost complementary pivoting algorithm is also addressed.

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